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Quantization and conformal properties of a generalized Calogero model

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Abstract. We analyze a generalization of the quantum Calogero model with the underlying conformal symmetry, paying special attention to the two-body model deformation. Owing to the underlying $SU(1,1)$ symmetry, we find that the analytic solutions of this model can be described within the scope of the Bargmann representation analysis, and we investigate its dynamical structure by constructing the corresponding Fock space realization. The analysis from the standpoint of supersymmetric quantum mechanics (SUSYQM), when applied to this problem, reveals that the model is also shape invariant. For a certain range of the system parameters, the two-body generalization of the Calogero model is shown to admit a one-parameter family of self-adjoint extensions, leading to inequivalent quantizations of the system.

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1 Introduction

The structure and application of the Calogero model [1–5] and its various descendants is a subject that receives much attention. Since these are examples of many-body exactly solvable models which appear in various contexts in physics as well as in mathematics [6–33], it is of considerable interest to find its generalizations which are exactly solvable and integrable. In particular, it is appealing to investigate whether some sort of modification of an exactly solvable model will affect its integrability. Dealing with such many-body problems in one dimension has appeared as particularly advantageous, since there exist several algebraic techniques that are applicable in this case, due to highly restrictive spatial degrees of freedom. It is also interesting to find out all possible boundary conditions that render the Hamiltonian of the system self-adjoint [34] and to analyze the nature of the corresponding spectrum.

In this paper we investigate one special class of deformation of the quantum Calogero model. As a prototype, we study the two-body model as a particularly convenient one for a complete elaboration of the dynamical structure of the problem, which is basically the same as for all many-body problems with the underlying $SU(1,1)$ symmetry. This specific case will be approached from more different directions that include the Bargmann representation analysis, the ladder operators formalism and the SUSYQM analysis which is based on the shape-invariance property of the model in question. We also study the self-adjoint extensions [34] of the two-body model and show that for certain values of the system parameters the model admits a one-parameter family of inequivalent quantizations. At the end, the analysis will be expanded to include the N-body case as well, with the result for the complete spectrum of the three-body case stated explicitly. The emphasis will particularly be given to the Bargmann representation approach which was put forward in [35–38] in the context of the general method for integration of the multi-species and multi-dimensional generalizations of the Calogero model. This approach relies heavily on the conformal invariance of the model under consideration.

2 Bargmann representation analysis

In a recent paper [35–38], a general procedure for integrating many-body quantum systems with the underlying $SU(1,1)$ symmetry was set up. This procedure is based on the fact that the N-body quantum system, possessing conformal symmetry, can be mapped onto the set of N harmonic oscillators in arbitrary dimensions and with a common frequency ω. In this paper we investigate the

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system described by the Hamiltonian

$$H = -\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + \frac{1}{2}\omega^2\left(x_1^2 + x_2^2\right) + \frac{\lambda}{2(x_1 - x_2)^2} + \frac{\mu}{2\left(x_1^2 + x_2^2\right)} \tag{1}$$

as a simple example of the model and techniques that were put forward in [35–38]. The above Hamiltonian clearly represents two interacting particles in one-dimensional space bounded by a harmonic force. For simplicity, the masses of the particles are set equal to 1 and $\hbar = 1$. It is convenient to introduce the following set of operators:

$$\begin{aligned} T_+ &= \frac{1}{2}\sum_{i=1}^{2} {x_i}^2 = \frac{1}{2}r^2, \\ T_- &= \frac{1}{2}\sum_{i=1}^{2}\frac{\partial^2}{\partial x_i^2} - V(x_1, x_2) \\ &= \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right) - V\,, \\ T_0 &= \frac{1}{2}\sum_{i=1}^{2} x_i\frac{\partial}{\partial x_i} + \frac{1}{2} = \frac{1}{2}r\frac{\partial}{\partial r} + \frac{1}{2}\,, \end{aligned} \tag{2}$$

with V given by

$$V = \frac{\lambda}{2(x_1 - x_2)^2} + \frac{\mu}{2\left(x_1^2 + x_2^2\right)}\,. \tag{3}$$

If we set $\mu = -\lambda$ in (1), we arrive at the model considered in [39]. The operators T_+, T_- , T_0 in (2) are expressed in terms of the polar coordinates $x_1 = r\sin\phi, x_2 = r\cos\phi$. A brief inspection of the Hamiltonian (1) reveals that it belongs to the class of conformally invariant systems [35–38], with the potential V being a real homogeneous function of order -2, i.e., satisfying the relation

$$\left[\sum_{i=1}^{2} x_i\frac{\partial}{\partial x_i}, V\right] = -2V\,. \tag{4}$$

Indeed, it is straightforward to show that the generators (2) satisfy the $SU(1,1)$ conformal algebra

$$[T_-, T_+] = 2T_0\,, \quad [T_0, T_\pm] = \pm T_\pm\,, \tag{5}$$

and that the Hamiltonian (1) can be represented in terms of these as $H = -T_- + \omega^2 T_+$. The problem of solving for the eigenstates and for the spectrum of the Hamiltonian (1), $H\psi_{n,k} = E_{n,k}\psi_{n,k}$, will now be transferred into the two-oscillator eigenvalue problem for the Hamiltonian $2\omega T_0$, which is usually referred to as a transition to a Bargmann representation. This can be achieved by applying the transformation

$$H = 2\omega S T_0 S^{-1}, \tag{6}$$

where

$$S = \mathrm{e}^{-\omega T_+}\mathrm{e}^{-\frac{1}{2\omega}T_-}. \tag{7}$$

After carrying out the transition to the Bargmann representation, we have to solve the eigenvalue problem for T_0, with an additional constraint

$$T_-\Delta_n = 0\,, \quad T_0\Delta_n = \frac{\epsilon_n}{2}\Delta_n\,, \quad n > 0\,. \tag{8}$$

As it stands, there are many eigenfunctions of T_0. In fact, every homogeneous function, no matter whether it is rational or irrational, is an eigenfunction of T_0. However, among all of them we have to pick up only those that are annihilated by T_-. These form an infinite number of vacua Δ_n, upon which the equidistant towers of states are built, with an elementary energy step 2ω between any two neighboring states. The ground state $\psi_{0,0}$ of (1) is required to be a square-integrable function. This will be the case if the ground-state energy $\omega\epsilon_0$, $\epsilon_0 > 0$, is larger than $\frac{\omega}{2}$, a condition which is connected with the existence of the critical point [40–45]. Then we can write $\psi_{0,0} = S\Delta_0$, where Δ_0 is a homogeneous function of the lowest degree and of the lowest energy $\omega\epsilon_0$. The other vacua Δ_n, $n > 0$, of higher degrees of homogeneity and with energies $\omega\epsilon_n$, are also mapped to $\psi_{n,0} = S\Delta_n$. As far as the relations (5) are concerned, the generators (2) can be viewed as creation and annihilation operators acting on the eigenstates of T_0. In respect of this, the excited states $\psi_{n,k}$ of (1) can be constructed as

$$\psi_{n,k} = S{T_+}^k\Delta_n\,, \quad k = 0, 1, 2, \ldots\,, \quad n \geq 0\,, \tag{9}$$

with energies $2\omega\left(k + \frac{\epsilon_n}{2}\right)$, and S given by (7). Thus, to conclude, we have all solutions grouped into equidistant towers of states based on $S\Delta_n, n \geq 0$, and for a given $n \geq 0$, the spectrum is equidistant with an elementary step 2ω.

It may be noted that the Casimir operator of the $SU(1,1)$ algebra (5), which is an element of the universal enveloping algebra, is given by

$$\mathcal{C} = T_0^2 - T_0 - T_+T_-\,, \tag{10}$$

which commutes with the Hamiltonian, with the operators on the right hand side being elements of the algebra (5). The given system is thus integrable with the Hamiltonian and the Casimir operator as the two conserved quantities. By using (8)–(10), we see that

$$\mathcal{C}\psi_{n,k} = \frac{\epsilon_n}{2}\left(\frac{\epsilon_n}{2} - 1\right)\psi_{n,k}\,. \tag{11}$$

Thus, it is evident that each tower of states built on the vacuum Δ_n provides an irreducible representation of the $SU(1,1)$ algebra classified by the eigenvalues of the Casimir operator $\mathcal{C}$ given in (11).

The aforementioned procedure will now be applied to the model Hamiltonian (1) to yield the eigenstates and the spectrum. First, we have to solve the equation $T_-\Delta_n = 0$, $n > 0$, for the potential (3),

$$\left(\frac{1}{2}\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}\frac{\partial^2}{\partial x_2^2} - \frac{\lambda}{2(x_1 - x_2)^2} - \frac{\mu}{2\left(x_1^2 + x_2^2\right)}\right)\Delta_n = 0\,. \tag{12}$$

In polar coordinates this equation reads

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}-\frac{\lambda}{r^2(1-\sin 2\phi)}-\frac{\mu}{r^2}\right)\Delta_n=0\,. \tag{13}$$

If we follow the lines of [39], by separating the variables as $\Delta_n=u(r)\Phi(\phi)$, (13) reduces to the pair of equations

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{C+\mu}{r^2}u=0\,, \tag{14}$$

$$\frac{\partial^2\Phi}{\partial\phi^2}-\left(\frac{\lambda}{1-\sin 2\phi}-C\right)\Phi=0\,. \tag{15}$$

The first one integrates to $u\sim r^{\sqrt{C+\mu}}$, while the second one appears to have physically acceptable solutions only for the special values of the coupling λ and the separation constant C. In order to find these solutions, we skip to the angular variable $\theta=\phi-\frac{\pi}{4}$, running through the interval $0\le\theta\le\pi$, and then factorize the function Φ,

$$\Phi_n(\theta)=(\sin\theta)^\nu f_n(\cos\theta)=(\sin\theta)^\nu f_n(x)\,, \tag{16}$$

where the variable $x=\cos\theta$ has been introduced and n is the quantum number labeling the vacua Δ_n. After inserting (16) into equation (15), we get the equation for the functions f_n, $n\ge 0$,

$$\begin{aligned}&\frac{\mathrm{d}^2 f_n(x)}{\mathrm{d}x^2}\sin^4\theta-(2\nu+1)x\frac{\mathrm{d}f_n(x)}{\mathrm{d}x}\sin^2\theta\\&\quad+\left(\left(\nu(\nu-1)-\frac{\lambda}{2}\right)x^2-\left(\nu+\frac{\lambda}{2}-C\right)\sin^2\theta\right)f_n(x)=0\,.\end{aligned} \tag{17}$$

Physically acceptable solutions to the above equation emerge for the special choices of the coupling and the separation constants λ and C, respectively,

$$\lambda=2\nu(\nu-1)\,,\quad C=(n+\nu)^2, \tag{18}$$

in which case (17) becomes the equation for the Jacobi polynomials $P_n^{(a,b)}$

$$(1-x^2)\frac{\mathrm{d}^2 f_n(x)}{\mathrm{d}x^2}-(2\nu+1)x\frac{\mathrm{d}f_n(x)}{\mathrm{d}x}+n(n+2\nu)f_n(x)=0\,. \tag{19}$$

Thus, as a solution to (19) we have $f_n(x)=P_n^{(a,a)}(x)$, with $a=\frac{2\nu-1}{2}$. Altogether, this gives us the following expression for the vacua in the Bargmann representation:

$$\begin{aligned}\Delta_n&=u\Phi_n=r^{\sqrt{C+\mu}}(\sin\theta)^\nu f_n(x)\\&=r^{\sqrt{(n+\nu)^2+\mu}}(\sin\theta)^\nu P_n^{(a,a)}(\cos\theta)\,,\\a&=\frac{2\nu-1}{2}\,.\end{aligned} \tag{20}$$

In view of (5), the excited states in the Bargmann representation are obtained as

$$T_+^k\Delta_n=\left(\frac{1}{2}r^2\right)^k r^{\sqrt{(n+\nu)^2+\mu}}(\sin\theta)^\nu P_n^{(a,a)}(\cos\theta)\,. \tag{21}$$

With the help of the transformation (7), we now transfer the results back (see (9)) to the original problem to obtain the eigenstates of (1),

$$\begin{aligned}\psi_{n,k}&=ST_+^k\Delta_n\\&\sim \mathrm{e}^{-\omega T_+}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\left(\frac{1}{2\omega}T_-\right)^l\\&\quad\times\left(r^{\sqrt{(n+\nu)^2+\mu}+2k}(\sin\theta)^\nu P_n^{(a,a)}(\cos\theta)\right)\,.\end{aligned} \tag{22}$$

For the moment, we shall not be bothered with the proportionality constants, because they are irrelevant for the discussion below. This is the reason why the similarity sign appears in (22) and in the majority of subsequent relations. Later on, when we consider a construction of the bosonic ladder operators, we shall have to take care of the normalization of the wave functions and every single proportionality factor will be important.

In calculating (22), we shall make use of the fact that the successive application of the operator T_- to the states (21) leads to an expression of the form

$$\begin{aligned}T_-^l\left(T_+^k\Delta_n\right)&\sim 2^{-l}2k\left(2k+2\sqrt{(n+\nu)^2+\mu}\right)\\&\quad\times(2k-2)\left(2k-2+2\sqrt{(n+\nu)^2+\mu}\right)\\&\quad\times(2k-4)\left(2k-4+2\sqrt{(n+\nu)^2+\mu}\right)\times\ldots\\&\quad\times(2k-2(l-1))\left(2k-2(l-1)+2\sqrt{(n+\nu)^2+\mu}\right)\\&\quad\times r^{\sqrt{(n+\nu)^2+\mu}+2k-2l}(\sin\theta)^\nu P_n^{(a,a)}(\cos\theta)\,.\end{aligned} \tag{23}$$

When l is equal to k, the above expression is, essentially, the Bargmann vacuum Δ_n, up to some constant. Due to the first equation of (8), the next application of T_- yields zero, causing the series in (22) to terminate, $T_-^l(T_+^k\Delta_n)=0$ for $l>k$. Relation (22) now becomes

$$\begin{aligned}\psi_{n,k}&\sim 2^{-k}\mathrm{e}^{-\frac{\omega}{2}r^2}r^{\sqrt{(n+\nu)^2+\mu}}(\sin\theta)^\nu P_n^{(a,a)}(\cos\theta)\\&\quad\times\left[2^k r^{2k}-\frac{2^{k+1}}{2\omega}k\left(k+\sqrt{(n+\nu)^2+\mu}\right)r^{2k-2}\right.\\&\quad+\frac{2^{k+2}}{(2\omega)^2 2!}k(k-1)\left(k+\sqrt{(n+\nu)^2+\mu}\right)\\&\quad\times\left(k-1+\sqrt{(n+\nu)^2+\mu}\right)r^{2k-4}\\&\quad+\ldots+(-1)^k\frac{2^{2k}}{(2\omega)^k k!}k(k-1)(k-2)\times\ldots\times 2\times 1\\&\quad\times\left(k+\sqrt{(n+\nu)^2+\mu}\right)\left(k-1+\sqrt{(n+\nu)^2+\mu}\right)\\&\quad\left.\times\ldots\times\left(1+\sqrt{(n+\nu)^2+\mu}\right)\right]\\&=\omega^{-k}\mathrm{e}^{-\frac{\omega}{2}r^2}r^{\sqrt{(n+\nu)^2+\mu}}(\sin\theta)^\nu P_n^{(a,a)}(\cos\theta)\\&\quad\times\sum_{l=0}^{k}(-1)^l(\omega r^2)^{k-l}\binom{k}{l}l!\binom{k+\sqrt{(n+\nu)^2+\mu}}{l}\,,\end{aligned} \tag{24}$$

where the binomial coefficients in (24) have the following meaning:

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} \equiv \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)}\,,$$

with Γ being the Euler gamma-function.

If we rearrange the sum in (24), we finally obtain the result

$$\begin{aligned}\psi_{n,k} &\sim \omega^{-k}\mathrm{e}^{-\frac{\omega}{2}r^2} r^{\sqrt{(n+\nu)^2+\mu}} (\sin\theta)^{\nu} P_n^{(a,a)}(\cos\theta) \\ &\quad\times \sum_{l=0}^{k} (-1)^{k-l} \frac{k!}{l!(k-l)!} \frac{\left(k+\sqrt{(n+\nu)^2+\mu}\right)!}{\left(l+\sqrt{(n+\nu)^2+\mu}\right)!} (\omega r^2)^l \\ &= (-1)^k k! \omega^{-k} \mathrm{e}^{-\frac{\omega}{2}r^2} r^{\sqrt{(n+\nu)^2+\mu}} (\sin\theta)^{\nu} P_n^{(a,a)}(\cos\theta) \\ &\quad\times L_k^{\sqrt{(n+\nu)^2+\mu}}(\omega r^2) \end{aligned} \tag{25}$$

for the eigenfunctions of the Hamiltonian (1). In obtaining this result, the sum in the second line of (25) is recognized [46, 47], up to the irrelevant factor (anyway, it enters the overall normalization constant of the wave function), as the power-series expansion for the associated Laguerre polynomials,

$$L_k^{\alpha}(x) = \sum_{m=0}^{k} \frac{(-1)^m}{m!} \binom{k+\alpha}{k-m} x^m, \quad \alpha > -1\,. \tag{26}$$

Considering the spectrum, as it is unaltered by the transformation (7), we conclude that the eigenenergies of (1) are nothing else but the degrees of homogeneity of the states in the Bargmann representation. When T_0 is applied to the Bargmann vacua (20), we obtain $\frac{1}{2}(\sqrt{(n+\nu)^2+\mu}+1)\Delta_n$ and according to the second one of (8), this has to be equal to $\frac{1}{2}\epsilon_n$. Knowing the dynamical structure of the problem considered (see Fig. 1), and anticipating that for a particular vacuum labeled by n, the energies of the neighboring excited states are separated by 2ω, the spectrum is easily

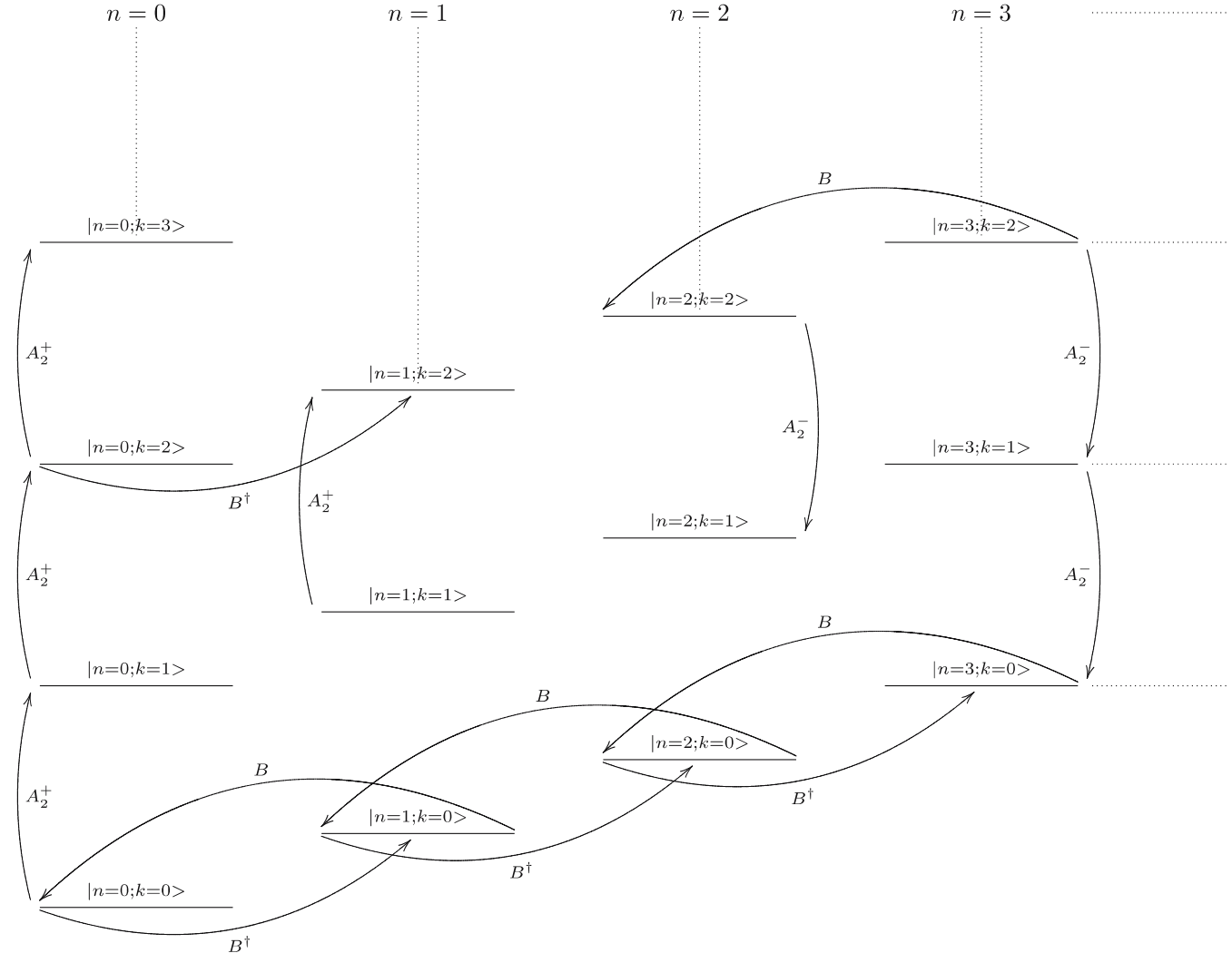

Fig. 1. Horizontal shifts in the Fock space of states are accomplished by the operators $B^\dagger$ and B, whereas the vertical shifts are provided by the pair of operators A_2^+ and A_2^-

found to be

$$E_{n,k} = \omega(\epsilon_n + 2k) = \omega\left(\sqrt{(n+\nu)^2+\mu}+1+2k\right), \quad (27)$$

with ν being determined by (18).

3 Ladder operators

As was said previously, in order to find bosonic ladder operators [48–51] related to the eigenfunctions we were in search for, the knowledge on the detailed form of the normalization factor is essential. Thus, after we utilize the normalization properties of the Jacobi and Laguerre polynomials, we are left with the normalized version of the eigenfunctions (25) of the model Hamiltonian (1),

$$\begin{aligned}\tilde{\psi}_{n,k} = &\sqrt{\frac{\omega^{\sqrt{(n+\nu)^2+\mu}+1} k!n!(n+\nu)\Gamma(n+2\nu)}{4^{\nu-1}\Gamma\left(\sqrt{(n+\nu)^2+\mu}+k+1\right)\Gamma(n+\nu+1/2)^2}}\\ &\times \mathrm{e}^{-\frac{\omega}{2}r^2} r^{\sqrt{(n+\nu)^2+\mu}} (\sin\theta)^{\nu} P_n^{(a,a)}(\cos\theta)\\ &\times L_k^{\sqrt{(n+\nu)^2+\mu}}(\omega r^2)\,. \end{aligned} \quad (28)$$

For brevity, we shall omit the tilde-symbol from the wave functions, and in the whole subsequent exposition it will be understood that they are normalized to unity. As we are searching for the operators that provide transitions between ground-state configurations of neighboring towers in Fig. 2, we basically look for the operators connecting Jacobi polynomials of successive order. A straightforward way of deducing the form of these ladder operators (in Fig. 2 they are designated by B and $B^\dagger$) is to invoke the recursive relations [46, 47] for Jacobi polynomials,

$$\begin{aligned}&(2n+a+b)(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(a,b)}(x)\\ &\quad = n[(a-b)-(2n+a+b)x]P_n^{(a,b)}(x)\\ &\qquad + 2(n+a)(n+b)P_{n-1}^{(a,b)}(x)\,, \end{aligned} \quad (29)$$

$$\begin{aligned}&2(n+1)(n+a+b+1)(2n+a+b)P_{n+1}^{(a,b)}(x)\\ &\quad = (2n+a+b+1)\\ &\qquad \times [(2n+a+b)(2n+a+b+2)x + a^2 - b^2]P_n^{(a,b)}(x)\\ &\qquad - 2(n+a)(n+b)(2n+a+b+2)P_{n-1}^{(a,b)}(x)\,. \end{aligned} \quad (30)$$

Using the recursive relations (29) and (30), we find recursive relations for the normalized ground-state energy

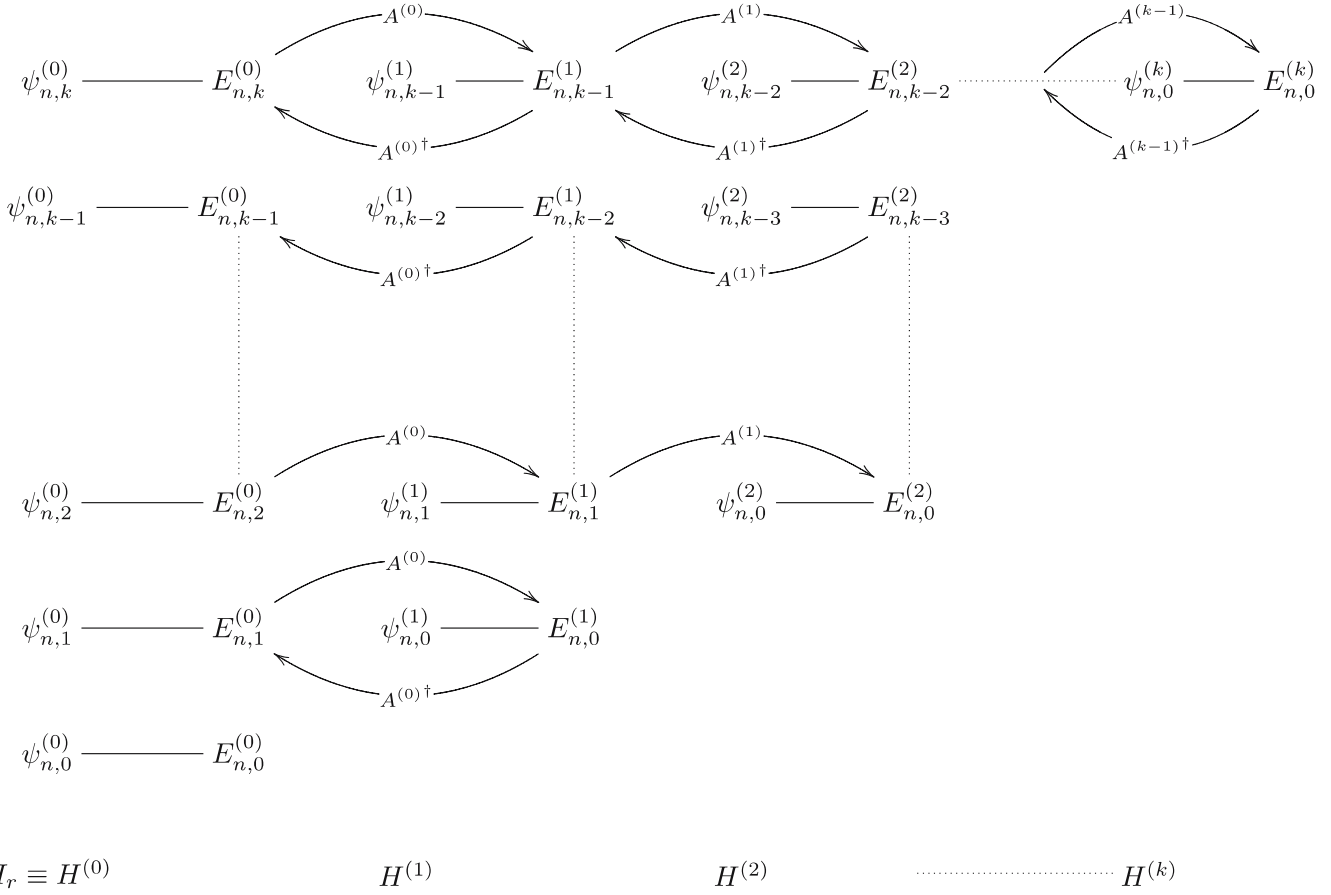

Fig. 2. A pattern for constructing the eigenstates of the model Hamiltonian H from the eigenstates of the supersymmetric partner Hamiltonians $H^{(i)}$

eigenfunctions $\psi_{n,0}$,

$$\left[(n+\nu)x+(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right]\psi_{n,0}$$
$$=(n+a)\sqrt{\frac{\omega^{\sqrt{(n+\nu)^2+\mu}+1}n!(n+\nu)\Gamma(n+2\nu)}{4^{\nu-1}\Gamma\left(\sqrt{(n+\nu)^2+\mu}+1\right)\Gamma(n+\nu+1/2)^2}}$$
$$\times \mathrm{e}^{-\frac{\omega}{2}r^2}r^{\sqrt{(n+\nu)^2+\mu}}(1-x^2)^{\nu/2}P_{n-1}^{(a,a)}(x)\,, \tag{31}$$

$$\left[(n-\nu+2a+1)x-(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right]\psi_{n,0}$$
$$=\frac{(n+1)(n+2a+1)}{n+a+1}$$
$$\times\sqrt{\frac{\omega^{\sqrt{(n+\nu)^2+\mu}+1}n!(n+\nu)\Gamma(n+2\nu)}{4^{\nu-1}\Gamma\left(\sqrt{(n+\nu)^2+\mu}+1\right)\Gamma(n+\nu+1/2)^2}}$$
$$\times \mathrm{e}^{-\frac{\omega}{2}r^2}r^{\sqrt{(n+\nu)^2+\mu}}(1-x^2)^{\nu/2}P_{n+1}^{(a,a)}(x)\,, \tag{32}$$

where $x=\cos\theta$ and $a=\nu-\frac{1}{2}$.

The ladder operators b and $b^\dagger$ that shift the neighboring vacua into each other can be read from the above recursions:

$$b=\left[x(N+\nu)+(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right]$$
$$\times\sqrt{\frac{\omega^{\sqrt{(N+\nu-1)^2+\mu}}}{\omega^{\sqrt{(N+\nu)^2+\mu}}}\frac{(N+\nu-1)}{(N+\nu)(N+2\nu-1)}}$$
$$\times\sqrt{\frac{\Gamma\left(\sqrt{(N+\nu)^2+\mu}+1\right)}{\Gamma\left(\sqrt{(N+\nu-1)^2+\mu}+1\right)}}\,, \tag{33}$$

$$b^\dagger=\left[x(N+\nu)-(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right]$$
$$\times\sqrt{\frac{\omega^{\sqrt{(N+\nu+1)^2+\mu}}}{\omega^{\sqrt{(N+\nu)^2+\mu}}}\frac{(N+\nu+1)}{(N+\nu)(N+2\nu)}}$$
$$\times\sqrt{\frac{\Gamma\left(\sqrt{(N+\nu)^2+\mu}+1\right)}{\Gamma\left(\sqrt{(N+\nu+1)^2+\mu}+1\right)}}\,, \tag{34}$$

with N being the number operator defined as

$$N\left[(1-x^2)^{\nu/2}P_n^{(a,a)}(x)\right]=n\left[(1-x^2)^{\nu/2}P_n^{(a,a)}(x)\right]. \tag{35}$$

The explicit form of this operator can be found with the help of (15) and looks like

$$N=\sqrt{L^2}-\nu\,,\quad L^2\equiv-\frac{\partial^2}{\partial\theta^2}+\frac{\lambda}{1-\cos 2\theta}\,. \tag{36}$$

Since N is only angular dependent, relation (35) can be straightforwardly extended to

$$N\psi_{n,0}=n\psi_{n,0}\,. \tag{37}$$

Straightforward calculation shows that the ladder operators b and $b^\dagger$ are bosonic,

$$[b,b^\dagger]=1\,, \tag{38}$$

together with

$$[N,b]=-b\,,\quad [N,b^\dagger]=b^\dagger, \tag{39}$$

resulting in the simple relation including the number operator $N=b^\dagger b$. However, they are still not the ladder operators we are looking for because they do not shift the neighboring eigenstates between each other, as it is readily seen from (31) and (32). To overcome this problem, it is convenient to consider a certain type of operators that are realized by means of the similarity transformation applied to the bosonic operators b and $b^\dagger$

$$B=r^{\sqrt{L^2+\mu}}br^{-\sqrt{L^2+\mu}}, \tag{40}$$
$$B^\dagger=r^{\sqrt{L^2+\mu}}b^\dagger r^{-\sqrt{L^2+\mu}}. \tag{41}$$

These operators have the desired properties, namely

$$B\psi_{n,0}=\sqrt{n}\psi_{n-1,0}\,,\quad B^\dagger\psi_{n,0}=\sqrt{n+1}\psi_{n+1,0}\,, \tag{42}$$

i.e., they are bosonic, satisfying

$$[B,B^\dagger]=1\,. \tag{43}$$

Owing to the similarity-transformation kind of relation between the operators b, $b^\dagger$ and B, $B^\dagger$, the latter pair retains the simple relation to the number operator, namely, $N=B^\dagger B$. Having established the form of the operators (40) and (41), we have succeeded to describe the horizontal shift in the Fock space of states, depicted in Fig. 1 by horizontal arrows. The vertical shift in Fig. 1 still remains to be described. However, it is easily accomplished by the transformed Bargmann representation creation and annihilation operators (2),

$$A_2^\pm=ST_\pm S^{-1}=\frac{1}{2}\left(\omega T_+ +\frac{1}{\omega}T_-\right)\mp T_0\,, \tag{44}$$

where T_+, T_-, T_0 are conformal generators (2) and S is the transformation (7). Of course, the state of the lowest energy $|0\rangle\equiv\psi_{n=0,k=0}$ is annihilated by both of the operators B and A_2^-. Now, the general Fock-space state of Fig. 1 is easily obtained by the successive application of the ladder operators (41) and (44) to the vacuum state $|0\rangle\equiv\psi_{n=0,k=0}$

$$\psi_{n,k}=\left(A_2^+\right)^k(B^\dagger)^n|0\rangle\,, \tag{45}$$

and in the coordinate representation it is explicitly described by (28).

4 SUSYQM analysis of the radial part

The model under consideration obviously possesses some supersymmetric features and can be approached from the point of view of the supersymmetric quantum mechanics [52–58]. To see this, we start with the Hamiltonian (1)

$$H = -\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)+\frac{1}{2}\omega^2\left(x_1^2+x_2^2\right) + \frac{\lambda}{2(x_1-x_2)^2}+\frac{\mu}{2\left(x_1^2+x_2^2\right)}\,. \tag{46}$$

By making a transition to polar coordinates and by separating the variables in the manner of $\Psi(r,\phi)=u(r)\Phi(\phi)$, the Schrödinger equation $H\Psi=E\Psi$ becomes

$$\frac{1}{u}r^2\frac{\partial^2 u}{\partial r^2}+\frac{1}{u}r\frac{\partial u}{\partial r}-\omega^2r^4-\mu+2Er^2=\frac{\lambda}{1-\sin 2\phi}-\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = C\,, \tag{47}$$

where C is the separation constant. This leads to the following separated equations:

$$\frac{1}{2}\left(-\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\omega^2r^2+\frac{C+\mu}{r^2}\right)u=Eu, \tag{48}$$

$$\left(-\frac{\partial^2}{\partial\phi^2}+\frac{\lambda}{1-\sin 2\phi}-\right)\Phi=C\Phi\,. \tag{49}$$

By introducing a new function $\psi(r)=\sqrt{r}u(r)$, (48) simplifies to

$$\frac{1}{2}\left(-\frac{\partial^2}{\partial r^2}+\omega^2r^2+\frac{C+\mu-\frac{1}{4}}{r^2}\right)\psi=E\psi\,. \tag{50}$$

From this equation we can immediately read the radial Hamilton

$$H_r=\frac{1}{2}\left(-\frac{\partial^2}{\partial r^2}+\omega^2r^2+\frac{C+\mu-\frac{1}{4}}{r^2}\right) =\frac{1}{2}\left(-\frac{\partial^2}{\partial r^2}+\omega^2r^2+\frac{\alpha^2-\frac{1}{4}}{r^2}\right), \tag{51}$$

where the parameter $\alpha=\sqrt{C+\mu}$ is introduced, with C determined by (18). The radial Hamiltonian (51) is shape invariant and this property can be verified by introducing the corresponding superpotential:

$$U=\omega r-\frac{\alpha+\frac{1}{2}}{r}, \tag{52}$$

and by factorizing the Hamiltonian (51) with the help of the operators $A^{(0)}$, ${A^{(0)}}^\dagger$ introduced in the following way:

$$A^{(0)}=\frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}r}+U\right)=\frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\omega r-\frac{\alpha+\frac{1}{2}}{r}\right), \tag{53}$$

$${A^{(0)}}^\dagger=\frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}r}+U\right)=\frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}r}+\omega r-\frac{\alpha+\frac{1}{2}}{r}\right). \tag{54}$$

Now we have

$$H_r={A^{(0)}}^\dagger A^{(0)}+e_0\,, \tag{55}$$

where e_0 is given by (27) as $e_0=E_{n,0}=\omega(\alpha+1)$. Let us now create a supersymmetric partner Hamiltonian $H_r^{(1)}$ of the Hamiltonian $H_r\equiv H_r^{(0)}$,

$$H_r^{(1)}=A^{(0)}{A^{(0)}}^\dagger+e_0\,. \tag{56}$$

The factorization of (56) in the way pursued in (55) can be achieved by introducing the operators

$$A^{(1)}=\frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\omega r-\frac{\alpha+\frac{3}{2}}{r}\right), \tag{57}$$

$${A^{(1)}}^\dagger=\frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}r}+\omega r-\frac{\alpha+\frac{3}{2}}{r}\right). \tag{58}$$

It is easy to check that the two sets of operators, (53), (54) and (57), (58), satisfy the SUSYQM shape-invariance condition

$$A^{(0)}{A^{(0)}}^\dagger={A^{(1)}}^\dagger A^{(1)}+e_1\,, \tag{59}$$

where $e_1=2\omega$.

A somewhat more detailed insight into the relationships among the quantities we are dealing with here is obtained by taking a look at Figs. 1 and 2. From Fig. 1, let us take an arbitrary tower of states, which, for the sake of argument, we can take to be the tower built upon the vacuum state labeled by n. This tower coincides identically with the zeroth tower shown in Fig. 2, with the other towers in Fig. 2 being the eigenstates of the corresponding supersymmetric partner Hamiltonians of the Hamiltonian (55). This correspondence is made obvious (see Fig. 2) by labeling each radial wave function by superscripts in parentheses. In this way, the superscript i put on a particular radial function designates that this function is the eigenfunction of the super partner Hamiltonian $H_r^{(i)}$, and so on. Each supersymmetric partner Hamiltonian $H_r^{(i)}$ has its own set of eigenstates $\psi_{n,k}^{(i)}$ and its own spectrum $E_{n,k}^{(i)}$, with n, i fixed and k being a nonnegative integer,

$$H_r^{(i)}\psi_{n,k}^{(i)}=E_{n,k}^{(i)}\psi_{n,k}^{(i)}\,,\quad k=0,1,\ldots \tag{60}$$

Note that the states from different towers but in the same horizontal line have the same energy and are transformed into each other by a simple pattern [52–56].

Having the whole picture set up, the basic vacuum (the state $\psi_{n,0}^{(0)}$) is obtained from the condition that the operator $A^{(0)}$ should annihilate it,

$$A^{(0)}\psi_{n,k=0}^{(0)}=0\,. \tag{61}$$

Equation (61) integrates to

$$\psi_{n,0}^{(0)}=r^{\alpha+\frac{1}{2}}\mathrm{e}^{-\frac{\omega}{2}r^2}. \tag{62}$$

In a similar way, the vacuum state of the $i=1$ tower is obtained as

$$A^{(1)}\psi^{(1)}_{n,k=0}=0\,, \tag{63}$$

leading to

$$\psi^{(1)}_{n,k=0}\sim r^{\alpha+\frac{3}{2}}\mathrm{e}^{-\frac{\omega}{2}r^2}. \tag{64}$$

The first excited state $\psi^{(0)}_{n,k=1}$ and the corresponding energy $E^{(0)}_{n,k=1}$ of the original Hamiltonian (51) follow as

$$\psi^{(0)}_{n,k=1}\sim A^{(0)\,\dagger}\psi^{(1)}_{n,k=0}\,, \tag{65}$$

giving

$$\begin{aligned}\psi^{(0)}_{n,k=1}&\sim \mathrm{e}^{-\frac{\omega}{2}r^2}r^{\alpha+\frac{1}{2}}[\omega r^2-(\alpha+1)]\\&=\mathrm{e}^{-\frac{\omega}{2}r^2}r^{\alpha+\frac{1}{2}}L^{\alpha}_1(\omega r^2)\,, \qquad (66)\\ E^{(0)}_{n,k=1}&=E^{(1)}_{n,k=0}=e_0+e_1=e_0+2\omega\,, \qquad (67)\end{aligned}$$

where $L^{\alpha}_k(\omega r^2)$ is the associated Laguerre polynomial and the last equation follows from the combined application of (56), (59) and (60).

By following the same line, we are led (see Fig. 2) to a pattern for constructing a general radial excitation together with the corresponding excitation energy. First, we introduce the pair of operators

$$A^{(k)}=\frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\omega r-\frac{\alpha+k+\frac{1}{2}}{r}\right), \tag{68}$$

$$A^{(k)\,\dagger}=\frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}r}+\omega r-\frac{\alpha+k+\frac{1}{2}}{r}\right). \tag{69}$$

The conjecture, now, is that after applying (69) j times successively,

$$\psi^{(k-j)}_{n,j}\sim A^{(k-j)\,\dagger}\dots A^{(k-2)\,\dagger}A^{(k-1)\,\dagger}\psi^{(k)}_{n,0}\,,$$

we get

$$\psi^{(k-j)}_{n,j}\sim r^{\alpha+\frac{1}{2}}\mathrm{e}^{-\frac{\omega}{2}r^2}r^{k-j}\sum_{s=0}^{j}(-1)^s\binom{j}{s}s!\binom{\alpha+k}{s}z^{j-s}, \tag{70}$$

where the variable $z=\omega r^2$ has been introduced. This expression can be proved by induction. For $j=1$, the expression (70) reduces to

$$\psi^{(k-1)}_{n,1}\sim r^{\alpha+\frac{1}{2}}\mathrm{e}^{-\frac{\omega}{2}r^2}(\omega r^{k+1}-(\alpha+k)r^{k-1})\,,$$

which coincides identically with $\psi^{(k-1)}_{n,1}\sim A^{(k-1)\,\dagger}\psi^{(k)}_{n,0}$. For the step of the induction, let us assume that (70) holds for j. Then we have

$$\begin{aligned}&\psi^{(k-(j+1))}_{n,j+1}\sim A^{(k-j-1)\,\dagger}\psi^{(k-j)}_{n,j}\\&=\frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}r}+\omega r-\frac{\alpha+k-j-\frac{1}{2}}{r}\right)\\&\quad\times\left(r^{\alpha+\frac{1}{2}}\mathrm{e}^{-\frac{\omega}{2}r^2}r^{k-j}\sum_{s=0}^{j}(-1)^s\binom{j}{s}s!\binom{\alpha+k}{s}z^{j-s}\right)\\&=\sqrt{2}r^{\alpha+\frac{1}{2}}\mathrm{e}^{-\frac{\omega}{2}r^2}r^{k-j-1}\\&\quad\times\left(\sum_{s=0}^{j}(-1)^s\binom{j}{s}s!\binom{\alpha+k}{s}\omega^{j-s+1}r^{2j-2s+2}\right.\\&\quad\left.-\sum_{s=0}^{j}(-1)^s\binom{j}{s}s!\binom{\alpha+k}{s}(\alpha+k-s)\omega^{j-s}r^{2j-2s}\right).\end{aligned}$$

By rearranging the sums and by using the properties of the binomial coefficients, the expression written above becomes

$$\begin{aligned}&\sqrt{2}r^{\alpha+\frac{1}{2}}\mathrm{e}^{-\frac{\omega}{2}r^2}r^{k-j-1}\\&\quad\times\left((\omega r^2)^{j+1}+(-1)^{j+1}j!\binom{\alpha+k}{j}(\alpha+k-j)\right.\\&\quad\left.+\sum_{s=1}^{j}(-1)^s\binom{j+1}{s}s!\binom{\alpha+k}{s}(\omega r^2)^{j+1-s}\right)\\&=\sqrt{2}r^{\alpha+\frac{1}{2}}\mathrm{e}^{-\frac{\omega}{2}r^2}r^{k-(j+1)}\\&\quad\times\sum_{s=0}^{j+1}(-1)^s\binom{j+1}{s}s!\binom{\alpha+k}{s}(\omega r^2)^{j+1-s},\end{aligned}$$

and this is exactly (70) for $j\to j+1$. Now that we have verified relation (70), the eigenstates of the radial Hamiltonian $H_r\equiv H^{(0)}_r$ follow directly from (70) by setting $j=k$,

$$\psi^{(0)}_{n,k}\sim r^{\alpha+\frac{1}{2}}\mathrm{e}^{-\frac{\omega}{2}r^2}\sum_{s=0}^{k}(-1)^s\binom{k}{s}s!\binom{\alpha+k}{s}z^{k-s}. \tag{71}$$

By rearranging the sum in the above expression, (71) is recognized as an expansion (26) for the associated Laguerre polynomials,

$$\psi^{(0)}_{n,k}\sim r^{\alpha+\frac{1}{2}}\mathrm{e}^{-\frac{\omega}{2}r^2}L^{\alpha}_k(\omega r^2)\,, \tag{72}$$

as it should be, to coincide with the results obtained previously by pursuing other methods. To complete the SUSYQM analysis, we turn onto the spectrum of the model. This analysis, of course, has to yield the same result as obtained before. While carrying out these considerations, one should note that the shape-invariance condition (59) holds at the general level, i.e.,

$$\begin{aligned}A^{(k-1)}A^{(k-1)\,\dagger}&=A^{(k)\,\dagger}A^{(k)}+e_k\,,\\ e_k&=2\omega\,,\quad k=1,2,3,\dots\,,\end{aligned} \tag{73}$$

as well as that the operators $A^{(k)}$ annihilate the corresponding vacua:

$$A^{(k)}\psi^{(k)}_{n,0}=0\,,\quad k=1,2,3,\dots\,. \tag{74}$$

Also, as Fig. 2 suggests, the energies of the states in different towers but in the same line are the same:

$$E^{(0)}_{n,k} = E^{(1)}_{n,k-1} = E^{(2)}_{n,k-2} = \ldots = E^{(k)}_{n,0}\,. \tag{75}$$

Following the construction pattern for the supersymmetric partner Hamiltonians $H^{(k)}_r$, their form is immediately deduced from the shape-invariance condition (73):

$$H^{(k)}_r = A^{(k)\dagger}A^{(k)} + \sum_{j=0}^{k} e_j\,, \quad k = 1, 2, 3, \ldots\,. \tag{76}$$

If we apply (76) to $\psi^{(k)}_{n,0}$, while simultaneously anticipating the relations (60), (74) and (75), we get the spectrum for H_r as

$$E^{(0)}_{n,k} = \sum_{j=0}^{k} e_j = e_0 + 2\omega k = \omega(2k+\alpha+1)\,, \tag{77}$$

coinciding with (27), as expected.

5 Self-adjoint extension

In this section we investigate the self-adjoint extensions [34] of the radial part of the Hamiltonian. Physically this means that we would find all possible boundary conditions for which the radial Hamiltonian is self-adjoint. This would be done by an appropriate analysis of the differential operator for the radial Hamiltonian. We shall see that the system admits a one-parameter family of self-adjoint extensions for certain values of the system parameters. We start with a rederivation of the energy eigenvalues discussed in Sect. 2 and this method would be carried over when we discuss self-adjoint extensions.

The Schrödinger equation obeyed by the Hamiltonian (1) of the system is given by (47). The operator involving the radial coordinate r in the l.h.s. of (47) satisfies the eigenvalue equation

$$\left[-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \omega^2 r^2 + \frac{\mu + C}{r^2}\right] u(r) = 2Eu(r)\,, \tag{78}$$

where $C = (n+\nu)^2$ as given in (18). We shall assume that $n \geq 0, \nu \geq 0$.

In order to proceed, we make the following transformations:

$$u(r) = r^{\alpha} \mathrm{e}^{-\frac{\omega r^2}{2}} \chi(r)\,, \quad \alpha = +\sqrt{\mu + C}\,, \tag{79}$$

$$t = \omega r^2, \tag{80}$$

and assume that $\mu + C > 0$. In these new variables, the radial equation (78) becomes

$$\left[t\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (\alpha + 1 - t)\frac{\mathrm{d}}{\mathrm{d}t} - \left(\frac{\alpha+1}{2} - \frac{E}{2\omega}\right)\right]\chi = 0\,, \tag{81}$$

whose solution is given by [46, 47]

$$\chi(r) = M\left(\frac{\alpha+1}{2} - \frac{E}{2\omega},\ \alpha+1,\ \omega r^2\right), \tag{82}$$

where M denotes the confluent hypergeometric function. The above solution in general consists of an infinite series. However, in order for the solution to be square integrable, this series must terminate, which happens when

$$\frac{\alpha+1}{2} - \frac{E}{2\omega} = -k\,, \tag{83}$$

where k is a positive integer. From this, and using the definitions of b and C, we obtain

$$E_{n,k} = \omega\left(\sqrt{(n+\nu)^2 + \mu} + 1 + 2k\right), \tag{84}$$

which is the same as (27). Also, substituting (83) in (82) and using the relation between the confluent hypergeometric and Laguerre functions [46, 47], we see that

$$\chi(r) = \frac{k!}{(\alpha+1)_k} L_k^{\sqrt{(n+\nu)^2+\mu}}(\omega r^2)\,, \tag{85}$$

where the symbol $(p)_n$ means

$$(p)_n = p(p+1)(p+2)\ldots(p+n-1)\,, \quad (p)_0 = 1\,. \tag{86}$$

The full solution of the radial equation is obtained by substituting (85) in (79).

We now consider the self-adjoint extensions of the operator O_r where

$$O_r = \frac{1}{2}\left[-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \omega^2 r^2 + \frac{\mu + C}{r^2}\right]. \tag{87}$$

To begin with, we briefly recall the salient points of von Neumann's theory of self-adjoint extensions [34].

Let T be an unbounded differential operator acting on a Hilbert space $\mathcal{H}$ and let $D(T)$ be the domain of T. The inner product of two elements, $\alpha, \beta \in \mathcal{H}$ is denoted by (α, β). Let $D(T^*)$ be the set of $\varphi \in \mathcal{H}$ for which there is a unique $\eta \in \mathcal{H}$ with $(T\xi, \varphi) = (\xi, \eta)\,\forall \xi \in D(T)$. For each such $\varphi \in D(T^*)$, we define $T^*\varphi = \eta$. The operator T^* then defines the adjoint of the operator T and $D(T^*)$ is the corresponding domain of the adjoint. The operator T is called symmetric or Hermitian if and only if $(T\varphi, \eta) = (\varphi, T\eta)\,\forall \varphi, \eta \in D(T)$. The operator T is called self-adjoint if and only if $T = T^*$ *and* $D(T) = D(T^*)$.

We now state the criterion to determine if a symmetric operator T is self-adjoint. For this purpose, let us define the deficiency subspaces $K_{\pm} \equiv \mathrm{Ker}(\mathrm{i} \mp T^*)$ and the deficiency indices $n_{\pm}(T) \equiv \dim[K_{\pm}]$. Then T falls in one of the following categories:

1) T is (essentially) self-adjoint if and only if $(n_+, n_-) = (0, 0)$.
2) T has self-adjoint extensions if and only if $n_+ = n_-$. There is a one-to-one correspondence between the self-

adjoint extensions of T and the unitary maps from K_+ into K_-.

3) If $n_+ \neq n_-$, then T has no self-adjoint extensions.

We now return to the discussion of the operator O_r. This is an unbounded differential operator defined in R^+. The operator O_r is a symmetric operator on the domain $D(O_r) \equiv \{\varphi(0) = \varphi'(0) = 0,\, \varphi,\, \varphi' \text{ absolutely continuous}, \varphi \in \mathrm{L}^2(r\,\mathrm{d}r)\}$. Next we would like to determine if O_r is self-adjoint. We shall focus on the case where $\alpha > 0$.

The deficiency indices $n_\pm$ are determined by the number of square-integrable solutions of the equations

$$O_r^* u_\pm(r) = \pm \mathrm{i} u_\pm(r)\,, \tag{88}$$

respectively, where O_r^* is the adjoint of O_r and the functions $u_\pm(r)$ span the deficiency subspaces $K_\pm$, respectively. Note that O_r^* is given by the same differential operator as O_r. From dimensional considerations we see that the r.h.s. of (88) should be multiplied by a constant with dimension of length^{-2}. We shall henceforth choose the magnitude of this constant to be unity by an appropriate choice of units.

Equation (88) can be written as

$$\left[t\frac{\mathrm{d}^2}{\mathrm{d}t^2} + (\alpha+1-t)\frac{\mathrm{d}}{\mathrm{d}t} - \left(\frac{\alpha+1}{2} \pm \frac{\mathrm{i}}{2\omega}\right)\right]\chi_\pm(r) = 0\,, \tag{89}$$

where

$$u_\pm(r) = r^\alpha \mathrm{e}^{-\frac{\omega r^2}{2}} \chi_\pm(r)\,. \tag{90}$$

The solutions $u_\pm(r)$ of (88) must be square integrable. We are thus led to choose the solutions given by

$$\chi_\pm(r) = U\big(g_\pm, \alpha+1, \omega r^2\big)\,, \tag{91}$$

where

$$U\big(g_\pm, \alpha+1, \omega r^2\big) = A\left[\frac{M\big(g_\pm, \alpha+1, \omega r^2\big)}{\Gamma(d_\pm)\Gamma(\alpha+1)} - (\omega r^2)^{-\alpha}\frac{M\big(d_\pm, 1-\alpha, \omega r^2\big)}{\Gamma(g_\pm)\Gamma(1-\alpha)}\right], \tag{92}$$

with $g_\pm = \frac{\alpha+1}{2} \mp \frac{\mathrm{i}}{2\omega}$, $d_\pm = \frac{1-\alpha}{2} \mp \frac{\mathrm{i}}{2\omega}$ and $A = \frac{\pi}{\sin(\pi(\alpha+1))}$. Consequently we obtain that

$$u_\pm(r) = r^\alpha \mathrm{e}^{-\frac{\omega r^2}{2}} U\big(g_\pm, \alpha+1, \omega r^2\big)\,. \tag{93}$$

The functions $u_\pm(r)$ are general solutions of (88) with the property that as $r \to \infty$, $u_\pm(r) \to 0$ sufficiently fast such that they are square integrable at infinity. In order to investigate the square integrability of the functions $u_\pm(r)$ near $r = 0$, first note that as $r \to 0$, $M(g_\pm, b+1, \omega r^2) \to 1$. Using these, we see that, as $r \to 0$,

$$|u_\pm|^2 r\,\mathrm{d}r \to \left[A_1 r^{(1+2\alpha)} + A_2 r + A_3 r^{(1-2\alpha)}\right]\mathrm{d}r\,, \tag{94}$$

where A_1, A_2 and A_3 are constants independent of r. From (94) it is clear that the functions $u_\pm(r)$ are not square integrable near $r = 0$ when $\alpha \geq 1$. Thus when $\alpha \geq 1$, we have the deficiency indices $n_+ = n_- = 0$ and the operator O_r is essentially self-adjoint in the domain $D(O_r)$. However, when $0 < \alpha < 1$, the functions $u_\pm(r)$ are square integrable near $r = 0$ and hence for the whole range of r. Thus, when $0 < \alpha < 1$, we have the deficiency indices $n_+ = n_- = 1$. In this case, according to von Neumann's theory, the operator O_r is not self-adjoint in the domain $D(O_r)$ but admits a one-parameter family of self-adjoint extensions which are labeled by $\mathrm{e}^{\mathrm{i}z}$, where $z \in R \pmod{2\pi}$. The domain of self-adjointness is given by $D_z(O_r) = D(O_r) \oplus \{a(u_+(r) + \mathrm{e}^{\mathrm{i}z} u_-(r))\}$, where a is an arbitrary complex number.

Before proceeding, let us discuss the nature of the parameter ranges for which the system is either essentially self-adjoint or admits self-adjoint extensions. From the condition on α, we see that the self-adjoint extension exists when

$$0 < \mu + (n+\nu)^2 < 1\,. \tag{95}$$

The eigenvalues of the angular equation are characterized by the integer $n \geq 0$. For each fixed value of n, (95) gives a region on the right half of the μ–ν plane for which self-adjoint extensions exist. Let us call this region on the μ–ν plane the nth band. It is evident that such bands, corresponding to different values of n, do not overlap. Consequently, if the values of the coupling constants μ and ν are fixed on the nth band, then the system will admit a self-adjoint extension only if the eigenvalue of the angular equation is taken as n. It is interesting to note that there exists a finite gap between any two consecutive bands labeled by the integers n and $n+1$. Since (95) is not satisfied for any eigenvalue of the angular equation within such band gaps and also for the region $\mu + \nu^2 > 1$, the system is essentially self-adjoint in these regions. Thus a band structure, with an infinite number of bands in the right half of the μ–ν plane, represents the parameter ranges where the system admits self-adjoint extensions. In Fig. 3 we have drawn the first two of such an infinite number of bands (i.e., the $n = 0$ and $n = 1$ band).

In order to discuss the solutions of the eigenvalue problem, first note that in the parameter range where the system is essentially self-adjoint, the spectrum has already been found before, in (82) and (84). We now proceed to find the spectrum of O_r in the domain $D_z(O_r)$ for the parameter range where the system admits self-adjoint extensions. To that end, first note that the solution of the eigenvalue equation (78) which is square integrable at infinity is given by

$$u(r) = B r^\alpha \mathrm{e}^{-\frac{\omega r^2}{2}} U\big(g, \alpha+1, \omega r^2\big)\,, \tag{96}$$

where $g = \frac{\alpha+1}{2} - \frac{E}{2\omega}$ and B is a constant. In the limit $r \to 0$, using (92), we get that

$$u(r) \to AB\left[\frac{r^\alpha}{\Gamma(d)\Gamma(\alpha+1)} - \frac{\omega^{-\alpha} r^{-\alpha}}{\Gamma(g)\Gamma(1-\alpha)}\right], \tag{97}$$

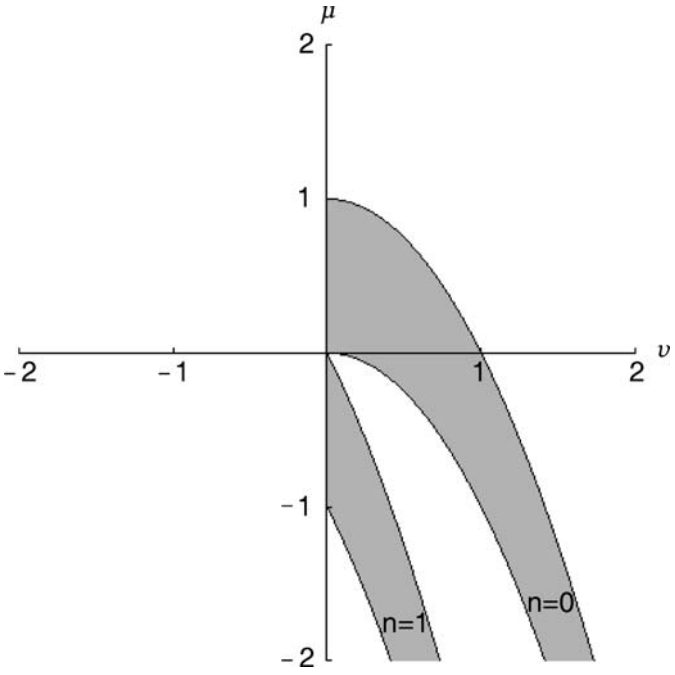

Fig. 3. The $n=0$ and $n=1$ bands, representing the parameter ranges where a self-adjoint extension is possible, are drawn. An infinite number of such bands exist in the right half of the μ–ν plane

where $d=\frac{1-\alpha}{2}-\frac{E}{2\omega}$. Again, as $r\to 0$, we see that

$$u_+(r)+\mathrm{e}^{\mathrm{i}z}u_-(r)\to A\left[\frac{r^{\alpha}}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(d_+)}+\frac{\mathrm{e}^{\mathrm{i}z}}{\Gamma(d_-)}\right)-\frac{\omega^{-\alpha}r^{-\alpha}}{\Gamma(1-\alpha)}\left(\frac{1}{\Gamma(g_+)}+\frac{\mathrm{e}^{\mathrm{i}z}}{\Gamma(g_-)}\right)\right]. \tag{98}$$

If $u(r)\in D_z(O_r)$, then by comparing the coefficients of different powers of r in (97) and (98), we get

$$f(E)\equiv\frac{\Gamma\left(\frac{1-\alpha}{2}-\frac{E}{2\omega}\right)}{\Gamma\left(\frac{1+\alpha}{2}-\frac{E}{2\omega}\right)}\frac{\rho_2\cos\left(\frac{z}{2}-\sigma_1\right)}{\rho_1\cos\left(\frac{z}{2}-\sigma_2\right)}, \tag{99}$$

where $\Gamma\left(\frac{1+\alpha}{2}+\frac{\mathrm{i}}{2\omega}\right)\equiv\rho_1\mathrm{e}^{\mathrm{i}\sigma_1}$ and $\Gamma\left(\frac{1-\alpha}{2}+\frac{\mathrm{i}}{2\omega}\right)\equiv\rho_2\mathrm{e}^{\mathrm{i}\sigma_2}$. For a given choice of the system parameters, (99) gives the energy eigenvalue E as a function of the self-adjoint parameter z. For a fixed set of system parameters, different choices of z lead to inequivalent quantization and to the spectrum for this model in the parameter range where it admits a self-adjoint extension. In general, the energy E cannot be calculated analytically and has to be obtained numerically by plotting (99), a sample of which is given in Fig. 4.

It is interesting to note that the usual N-body Calogero model with the confining interaction leads to a similar radial Hamiltonian, whose self-adjoint extension has been studied before [33]. An alternative treatment of the radial problem in the SUSYQM framework, in terms of the self-adjoint extension of the supercharges (first order operators), has been given in [59]. Although the radial operators in these works share the formal structure, the associated physical interpretations vary, e.g. the appearance of the band structure in the parameter space of the present model (as shown in Fig. 3) is absent in the case of the usual Calogero model with confining interaction.

We conclude this section with the following observations.

1. If $\Gamma\left(\frac{1+\alpha}{2}-\frac{E}{2\omega}\right)=\infty$, we obtain $E_k=\omega(2k+\alpha+1)$, where k is a positive integer. For this to happen, the self-adjoint extension parameter z must take the value $\pi+2\sigma_1$, and for this choice of z we recover the usual eigenvalues of this system.
2. For any other choice of z, the spectrum of O_r must be obtained numerically, and it is seen that the corresponding spectrum is not equispaced in the quantum

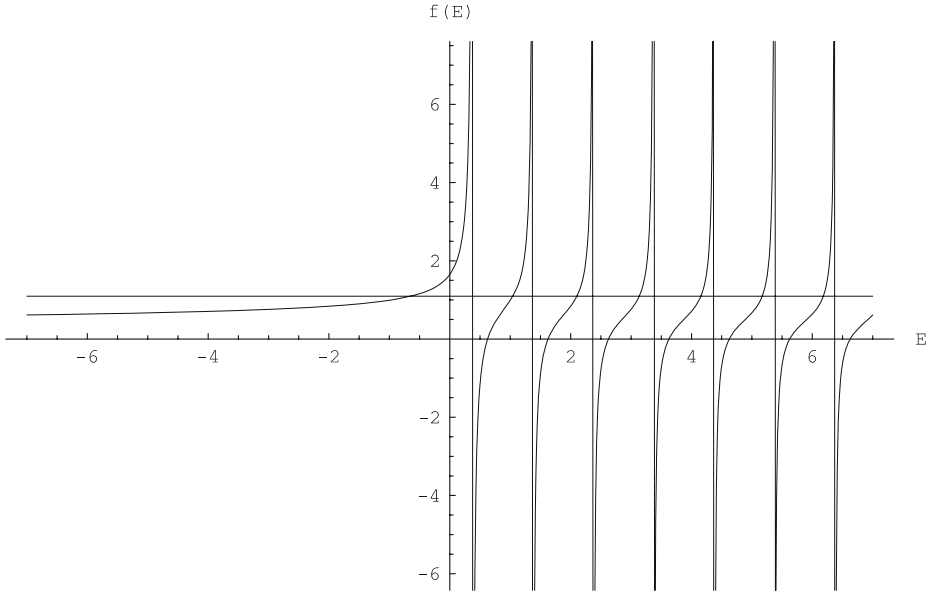

Fig. 4. A plot of (99) using Mathematica with $\omega=0.25$, $\alpha=0.25$ and $z=-1.5$. The *horizontal straight line* corresponds to the value of the r.h.s. of (99)

number k. Moreover, generically there is a single negative energy solution. This is due to the fact that for the parameter range where this model admits a self-adjoint extension, and for a generic value of z, $SU(1,1)$ can no longer be implemented as a spectrum generating algebra.

3. Although we have assumed that $\mu + C > 0$, in general this restriction is not necessary. If this is equal to zero, the system can still be shown to admit a one-parameter family of self-adjoint extensions by using an analysis similar to the one presented here. Also, if $\mu + C$ is strongly negative, then this problem might require a renormalization [60–62], which will be studied elsewhere.

6 Generalization to the N-body case

Following the same procedure as for the two-body case, let us now investigate some important features of the N-body case. In particular, we are interested in the general structure of the spectrum which basically has the same structure as that shown in Fig. 2, except that for the N-body case, the Bargmann vacua depend on $N-1$ quantum numbers. The problem under consideration is described by the Hamiltonian

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\omega^2\sum_{i=1}^{N}x_i^2 + \frac{\lambda}{2}\sum_{i<j}\frac{1}{(x_i-x_j)^2} + \frac{\mu}{2\sum_{i=1}^{N}x_i^2}\,, \tag{100}$$

with $\lambda = 2\nu(\nu-1)$. In [63] some other deformations of the basic inverse-square interparticle-distance model are considered, but these models do not possess the $SU(1,1)$ symmetry. This makes them unsuitable for the treatment within the scope of the techniques based on the transition to the Bargmann representation. Owing to the underlying conformal structure of the model (100), the spectrum and the eigenstates can be deduced by employing the same techniques as before. These are based on the fact that the Hamiltonian (100) has some special favorable properties which allow us to link it with the set of decoupled oscillators that is much easier to handle with. In order to do this, we introduce the set of operators

$$\begin{aligned} T_+ &= \frac{1}{2}\sum_{i=1}^{N}x_i^2\,,\\ T_- &= \frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} - \frac{\lambda}{2}\sum_{i<j}\frac{1}{(x_i-x_j)^2} - \frac{\mu}{2\sum_{i=1}^{N}x_i^2}\,,\\ T_0 &= \frac{1}{2}\sum_{i=1}^{N}x_i\frac{\partial}{\partial x_i} + \frac{N}{4}\,, \end{aligned} \tag{101}$$

satisfying (5). The connection of the Hamiltonian (100) with the set of decoupled oscillators [64–68], described by the operator T_0, is provided through the transformation (7), where now T_+ and T_- are from (101). As a side remark, let us note that this is exactly what we mean when we are referring to the transition to the Bargmann representation. After this transition we are left with the equation $T_-\Delta = 0$, which has to be solved in order to find all Bargmann vacua. Of course, they will now depend on $N-1$ quantum numbers. The remaining quantum number required for the complete determination of the eigenfunctions (eigenfunctions of (100) are fully specified with N quantum numbers) will describe excitations within each tower of states built over the corresponding ground state (see Fig. 2). These excitations are governed by the collective relative radial motion of the particles and are a universal feature [35–38, 40–42] of all models possessing the underlying $SU(1,1)$ symmetry.

In finding the solution to the equation $T_-\Delta = 0$, one can make a factorization

$$\Delta = \prod_{i<j}(x_i - x_j)^{\nu}\phi\,, \tag{102}$$

to obtain an equation for ϕ,

$$\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}\phi + \nu\sum_{i<j}\frac{1}{x_i-x_j}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)\phi - \frac{\mu}{2\sum_{i=1}^{N}x_i^2}\phi = 0\,. \tag{103}$$

One solution to this equation is of the form

$$\phi \sim \left(\sum_{i=1}^{N}x_i^2\right)^{\beta}\,, \tag{104}$$

with

$$\beta = \frac{\nu N - \nu N^2 - N + 2 + \sqrt{(N-2+\nu N^2-\nu N)^2 + 4\mu}}{4}\,. \tag{105}$$

It is clear that for each ordering of particles the solution (104) does not have nodes and hence it is the solution for the ground state. As far as we are interested in the energy and the wave function of the ground state and all vertical (see Fig. 1) excitations over it, the solution (104) is all we need to deduce the spectrum and wave functions over the lowest lying vacuum in Fig. 1. In order to obtain the ground state of the model Hamiltonian (100) and to construct all its wave functions $\psi_{0,k,0,\dots,0}$ belonging to the first tower of Fig. 1, we have to use the transformation (7) to return back from the Bargmann representation describing the set of decoupled oscillators,

$$\begin{aligned} \psi_{0,k,0,\dots,0} &\sim \mathrm{e}^{-\omega T_+}\mathrm{e}^{-\frac{1}{2\omega}T_-}\left((T_+)^k\Delta\right)\\ &= \frac{(-1)^k k!}{(2\omega)^k}\left(\sum_{i=1}^{N}x_i^2\right)^{\beta}\\ &\quad\times\prod_{i<j}(x_i-x_j)^{\nu}\mathrm{e}^{-\omega T_+}L_k^{\alpha}(2\omega T_+)\,, \end{aligned} \tag{106}$$

where $L_k^{\alpha}(2\omega T_+)$ are the associated Laguerre polynomials with

$$\alpha = \frac{1}{2}\sqrt{(N-2+\nu N^2-\nu N)^2+4\mu}\,. \tag{107}$$

The corresponding part of the spectrum follows from the equation $T_0\Delta = \frac{\epsilon_0}{2}\Delta$, with T_0 and Δ having the form of (101) and (102), respectively, and is given by

$$\begin{aligned} E_{0,k,0,\dots,0} &= \omega(\epsilon_{0,0,\dots,0}+2k) \\ &= \omega\left(1+\frac{1}{2}\sqrt{(N-2+\nu N^2-\nu N)^2+4\mu}+2k\right). \end{aligned} \tag{108}$$

Here ϵ_0 is an abbreviation for $\epsilon_{0,0,\dots,0}$ designating the energy of the ground state, that is, the energy of the lowest of the Bargmann vacua which were seen to depend on $N-1$ quantum numbers.

Of course, there still remains the problem of finding all solutions to (103). Obviously, there are many solutions to (103) and each of them would define one particular Bargmann vacuum. Once we find all these solutions, we have completed the task of integrating the model Hamiltonian (100) because all excited states built over some particular solution $\Delta_{n_1,n_3,\dots,n_N}$ to (103) are described by an associated Laguerre polynomial in the radial variable $2\omega T_+$. So, the associated Laguerre polynomial will appear to describe excitations in each tower of states shown in Fig. 1 and, as we have seen, this is a common feature for all conformally invariant models.

If we look at the problem of integration of (103) more closely, we are naturally led to the question on the superintegrability [69] of the Hamiltonian (100). This question is closely related [69] to the problem of the separability and even multi-separability of the corresponding Schrödinger equation. Namely, a superintegrable system is one that admits more integrals of motion than it has degrees of freedom, and if it is characterized by a complete set of commuting quadratic integrals of motion, then it is also multiseparable. This means that its Schrödinger equation (i.e. the Hamilton–Jacobi equation in the classical case) allows for the separation of variables in more than one orthogonal system of coordinates. Since it is known that the N-body Calogero model [69] is superintegrable, it would be of interest to see whether the extra term in (100) changes anything in this respect. To get the answer to this question, it is most convenient to consider the three-body simplification. If we write the determining condition for the Bargmann vacua, $T_-\Delta = 0$, in spherical coordinates and separate the radial part from the angular one, we are confronted with the set of relations

$$\frac{\partial^2}{\partial r^2}u+\frac{2}{r}\frac{\partial}{\partial r}u-\frac{C+\mu}{r^2}u=0\,, \tag{109}$$

$$\begin{aligned} &\lambda(G(\theta,\phi)-C)F_{n,m}-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_{n,m}}{\partial\theta}\right) \\ &-\frac{1}{\sin^2\theta}\frac{\partial^2 F_{n,m}}{\partial\phi^2}=0\,, \end{aligned} \tag{110}$$

where C is the integration constant and

$$\begin{aligned} G(\theta,\phi) &= \frac{1}{\sin^2\theta(1-\sin 2\phi)}+\frac{1}{(\sin\theta\cos\phi-\cos\theta)^2} \\ &+\frac{1}{(\sin\theta\sin\phi-\cos\theta)^2}\,. \end{aligned} \tag{111}$$

In the above procedure it is understood that the Bargmann vacua depend on two quantum numbers, as is readily expected from the general consideration, and they are separated as

$$\Delta_{n,m} = u(r)F_{n,m}(\theta,\phi)\,. \tag{112}$$

While (109) is easily integrated to $u\sim\frac{1}{\sqrt{r}}r^{1/2}\sqrt{1+4(C+\mu)}$, the angular equation remains a much more difficult problem to solve. As has already been said, this problem amounts to the question of the superintegrability of the three-body version of the Hamiltonian (100) and consequently to the problem of the separability of (110). Nevertheless, we can deduce the whole spectrum by relying only on the expression (112) and on the fact that the spectrum has to become a Calogero one in the limit when the parameter μ approaches zero. With these observations in mind, the whole spectrum for the three-body variant of the Hamiltonian (100) is readily found to be

$$\begin{aligned} E_{n,k,m} &= \omega\Bigg(1+2k \\ &+\frac{1}{2}\sqrt{1+4(3\nu+n+3m)(1+3\nu+n+3m)+4\mu}\Bigg). \end{aligned} \tag{113}$$

7 Conclusion

We have considered a conformally invariant deformation of the quantum Calogero model with the special emphasis on the deformation of the two-body model. Owing to the fact that the model under consideration possesses $SU(1,1)$ symmetry, we were able to apply the techniques based on the correspondence between some particular conformally invariant model and the set of decoupled oscillators, the procedure which is referred to as the Bargmann representation analysis. This has provided us with the possibility to construct the creation and annihilation operators acting on the Fock space, thereby allowing us to investigate the dynamical structure of the problem.

We have also analyzed the self-adjoint extensions of the radial Hamiltonian of the two-body problem and have found the region in the parameter space where the system admits a one-parameter family of self-adjoint extensions. In the situations where the system admits self-adjoint extensions, $SU(1,1)$ can no longer in general be implemented as the spectrum generating algebra, and the corresponding spectrum is not equispaced in the quantum number k. However, for a special value of the self-adjoint extension parameter, $SU(1,1)$ can be recovered as the spectrum gen-

erating algebra, in which case the spectrum becomes equispaced in k. It is plausible that the angular part of the Hamiltonian given in (49) might also admit self-adjoint extensions, analogous to the angular Hamiltonian of the three-body Calogero model as discussed in [70].

Finally, we have carried out the generalization to include the N-body problem as well. Here we have found the ground state, which is the lowest of all Bargmann vacua, and excitations over it, together with the corresponding spectrum. Also, we have found that all excitations are basically the same, all of them having the origin in the collective relative radial motion of the particles, and they are described by the associated Laguerre polynomials in the collective radial variable. It appears that this is, in fact, a common feature of all models possessing the underlying $SU(1,1)$ symmetry. The only problem, but in no case a simple one, that has been left is to find all Bargmann vacua, that is, to find all solutions to (103). We have seen that this problem is closely related to the problem of the superintegrability of the Hamiltonian in question. To somehow clear up the situation, we have made the simplification to consider the three-body problem. Although we have not found how all Bargmann vacua look like in the three-body problem, we were able to deduce the complete spectrum for this case. All that was needed was the information on the underlying conformal invariance of the model and the explicit form of the radial part of the Bargmann vacua wave functions, together with noting that the Calogero spectrum has to be reproduced in a smooth limit when the deformation parameter tends to zero. We hope that the issues that are left unresolved, as is the case with the structure of the Bargmann vacua in a general N-body problem, will be addressed in the near future, at least for the three-body case.

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